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# LRO in lattice systems of linear oscillators with strong bilinear pair nearest-neighbour interaction

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**Abstract.** For systems of one-component interacting oscillators on a $d$-dimensional lattice, $d > 1$, whose potential energy besides a large nearest-neighbour ferromagnetic bilinear term contains a small general translation-invariant term, the existence of a ferromagnetic long-range order for two-valued lattice spins, equal to the sign of the oscillator variables, is established with the help of a Peierls-type contour bound. The Ruelle superstability bound is used to derive the contour bound.

## 1. Introduction and main result

Let us consider two systems of one-dimensional oscillators on the $d$-dimensional lattice $\mathbb{Z}^d$, $d > 1$ with the potential energies depending on oscillator variables, labelled by a set $\Lambda$ with finite cardinality $|\Lambda|$ and with free boundary conditions, i.e. the case when there are no oscillators in $\Lambda^c = \mathbb{Z}^d \backslash \Lambda$,

$$
\begin{aligned}
U(q_\Lambda) &= \sum_{x,y\in\Lambda} u_{x-y}(q_x, q_y) + U'(q_\Lambda) \\
u_{x-y}(q_x, q_y) &= \delta_{|x-y|,1}\left[\tfrac{1}{2}(u(q_x) + u(q_y)) - g q_x q_y\right]
\end{aligned}
\tag{1.1}
$$

$$
U(q_\Lambda) = \sum_{x\in\Lambda} 2d(u(q_x) - g q_x^2) + \tfrac{1}{2} g \sum_{x,y\in\Lambda,|x-y|=1} (q_x - q_y)^2 + U'(q_\Lambda). \tag{1.2}
$$

Here $q_x$ is the oscillator coordinate taking a value in $\mathbb{R}$, $q_X = (q_x, x \in X)$, the one-particle potential (external field) $u$ is a bounded from below even polynomial of degree $\deg u = 2n$, $U'$ is an even translation-invariant function such that $U$ satisfies the superstability and regularity conditions, $|x|$ is the Euclidean norm of the integer-valued vector $x$, $\delta_{x,y} = 1, x = y; = 0, x \neq y$.

Let us rewrite the expression for the potential energy (1.1) with the interaction part represented in a translation-invariant form, using the equality $q_x q_y = \frac{1}{2}[q_x^2 + q_y^2 - (q_x - q_y)^2]$ and the fact that a lattice site $x$ has $2d$ (or $2d-1$) nearest neighbours if $x \notin \partial\Lambda$, i.e. it does not belong to the boundary of $\Lambda$ (or $x \in \partial\Lambda$),

$$
U(q_\Lambda) = \sum_{x\in\Lambda} 2d(u(q_x) - g q_x^2) + \tfrac{1}{2} g \sum_{\substack{x,y\in\Lambda \\ |x-y|=1}} (q_x - q_y)^2 + U'(q_\Lambda) - \sum_{x\in\partial\Lambda} (u(q_x) - g q_x^2).
$$

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Let $\langle\ \rangle_\Lambda$, $\langle\ \rangle$ denote the Gibbs average for the system confined to $\Lambda$ and the system in the thermodynamic limit, i.e. $\Lambda = \mathbb{Z}^d$, respectively,

$$\langle F_X\rangle_\Lambda = Z_\Lambda^{-1}\int F_X(q_X)\,\mathrm{e}^{-\beta U(q_\Lambda)}\,\mathrm{d}q_\Lambda = \int F_X(q_X)\rho^\Lambda(q_X)\,\mathrm{d}q_X$$

$$\rho^\Lambda(q_X) = Z_\Lambda^{-1}\int \mathrm{e}^{-\beta U(q_\Lambda)}\,\mathrm{d}q_{\Lambda\backslash X} \qquad Z_\Lambda = \int \mathrm{e}^{-\beta U(q_\Lambda)}\,\mathrm{d}q_\Lambda.$$

Here the integration is performed over $R^{|\Lambda|}$ and $\rho^\Lambda$ are the correlation functions. By $\rho$ we will denote the correlation functions in the thermodynamic limit.

For several classes of ferromagnetic systems, differing from (1.1) and (1.2) by boundary terms, or systems with the nearest-neighbour (NN) pair interaction, which are included in the systems (1.1), the following Peierls-type (contour) bound was derived in [BF, FL]:

$$\left\langle \prod_{\langle x,x'\rangle\in\Gamma} \chi_x^+\chi_{x'}^-\right\rangle_\Lambda \leqslant \mathrm{e}^{-E|\Gamma|} \tag{1.3}$$

where $E$ is an increasing function of $\beta$ independent of $\Lambda$ , $\Gamma$ is a set of nearest neighbours, $|\Gamma|$ is the number of nearest neighbours in $\Gamma$,

$$\chi_x^+ = \chi_{(0,\infty)}(q_x) \qquad \chi_x^- = \chi_{(-\infty,0)}(q_x)$$

where $\chi_{(a,b)}$ is the characteristic function of the open interval $(a, b)$.

If one puts $s_x = \operatorname{sign} q_x$, then taking into account that $\chi_x^{+(-)} = \frac{1}{2}[1 + (-)s_x]$ one obtains

$$4\langle\chi_x^+\chi_y^-\rangle_\Lambda = 1 + \langle s_x\rangle_\Lambda - \langle s_y\rangle_\Lambda - \langle s_x s_y\rangle_\Lambda.$$

Since the systems are invariant under the transformation of changing signs of the oscillator variables, we have

$$\langle s_x s_y\rangle_\Lambda = 1 - 4\langle\chi_x^+\chi_y^-\rangle_\Lambda.$$

Now in order to prove the ferromagnetic long-range order (LRO) for the spins $s_x$, one has to show that the average of the right-hand side of the equality is strictly less than $\frac{1}{4}$. This can be proved with the aid of the following lemma [GJS, FL].

**Lemma 1.1.** *If the bound (1.3) holds and* $\mathrm{e}^{-E}$ *is sufficiently small then there exist positive numbers* $a$, $a'$ *such that*

$$\langle\chi_x^+\chi_y^-\rangle \leqslant a'\mathrm{e}^{-aE}. \tag{1.4}$$

*So, if one shows that* $E$ *can be made arbitrary large while increasing* $g$ *or* $\beta$*, then the LRO for the above spins will be proved.*

In [BF] the derivation of (1.3) is elegant and is based on the application of the Griffiths and Jensen inequalities. In [FL] it is derived with the aid of a chessboard bound following from the reflection positivity of NN interaction (see, also, [BW, Sh, Si]).

In this paper we prove the ferromagnetic LRO with the help of (1.3) for the systems in which the interaction is neither ferromagnetic nor NN, but is essentially ferromagnetic for sufficiently large $g$ (see remark 6).

We establish (1.3) for the simplest even polynomial $u(q) = \eta q^{2n} + u^1(q)$, $\deg u^1 \leqslant n$, with the help of the Ruelle superstability bound [R] and show that $E$ in (1.3) is positive and growing for increasing $g$, or more precisely

$$E = e_0 - \ln\left(4\sqrt{2\pi e e_0}\right) - E_0 \qquad e_0 = \left[g^n(\eta n)^{-1}\right]^{1/(2n-2)} \tag{1.5}$$

where $E_0$ depends on $g$, $\beta$ and is a bounded function of $g$, found from the superstability bound (for the rescaled and translated correlation functions). $e_0$ is the minimum of the simplest polynomial one-particle rescaled potential $u_g^0(q) = 2d(g^{-n}\eta q^{2n} - q^2)$.

The proposed technique is based on precise knowledge of how the constant, defining $E_0$ in the superstability bound, depends on the potential energy (theorem 2.1). It is inspired by the technique used for ferromagnetic quantum oscillator systems with the potential energy almost coinciding with (1.2) for $n = 2$ and $U'$, expressed through an infinite-range quadratic translation-invariant pair ferromagnetic potential, in [AKR]. In this paper a small parameter, appearing in the one-particle potential, is not associated with the magnitude of NN ferromagnetic interaction.

Our approach stresses the necessity of considering a large-magnitude NN ferromagnetic interaction (see remark 5).

Let us put $\|C\|_1 = \sum_x |C_x|$, where the summation is performed over the $d$-dimensional lattice and $C$ is a complex-valued function on $\mathbb{Z}^d$.

**Theorem 1.1.** *Let the potential energy $U(q_\Lambda)$ of the one-component oscillator system be given by (1.1) and (1.2), where $u(q) = \eta q^{2n} + u^1(q)$, $u^1(q) = \sum_{s=1}^{[n'/2]} u^1_{,s} q^{2s}$, $n' \leqslant n$, $u^1_{,s} \in \mathbb{R}$, $n > 1$, with $[l]$ being the integer part of $l$.*

*Also let $U'$ be a translation-invariant and an even function such that the superstability and regularity conditions hold for it*

$$U'(q_\Lambda) \geqslant -\sum_{x\in\Lambda} \left[Bv^0(q_x) + B'\right]$$

$$\left|W'(q_{X_1;q_{X_2}})\right| = \left|U'(q_{X_1\cup X_2}) - U'(q_{X_1}) - U'(q_{X_2})\right| \leqslant \sum_{\substack{x\in X_1\\ y\in X_2}} (v^0(q_x) + v^0(q_y))\Psi'_{|x-y|} \qquad \|\Psi'\|_1 < \infty$$

$$v^0(q) = \sum_{s=1}^{n^0} g^{l_s} q^{2s} \qquad n^0 < n$$

*where $B, B', \Psi'_{|x|} \geqslant 0$ do not depend on $g$, oscillator variables and $\Lambda$; $l_s \leqslant 0$ for non-positive $U'$ in the regularity condition for $U'$; $l_s < s$ for $s > 1$ and $l_1 \leqslant 1$ only for non-negative $U'$.*

*Then there is the ferromagnetic LRO for the spins $s_x$ for sufficiently large $g$: $g \gg 1$, i.e. $\langle s_x s_y \rangle > 0$.*

Since $s_x$ are scale invariant and their average is not changed after rescaling of oscillator variables, we can deal with the variables rescaled by $g^{-1/2}$ and the potential energy $U_g$,

$$U_g(q_\Lambda) = \sum_{x\in\Lambda} u_g(q_x) + \frac{1}{2}\sum_{\substack{|x-y|=1\\ x,y\in\Lambda}} (q_x - q_y)^2 + U'(g^{-1/2}q_\Lambda) - (2d)^{-1}\sum_{x\in\partial\Lambda} u_g(q_x) \tag{1.6}$$

where

$$u_g(q) = 2d(u(g^{-1/2}q) - q^2) \qquad g \geqslant 1.$$

The rescaled expression for (1.2) does not contain the last term in the right-hand side of (1.6).

The correlation functions generated by $U_g$ will be denoted by $\rho_g$.

The main idea of the proof originates from the inequality ($g \geqslant 1$),

$$\left\langle \prod_{\langle x,x'\rangle\in\Gamma} \chi_x^+ \chi_{x'}^- \right\rangle_\Lambda \leqslant \left(4\sqrt{2\pi e e_0}\right)^{|\Gamma|} \mathrm{e}^{-e_0|\Gamma|} \langle \mathrm{e}^{Q_{g,\Gamma}} \rangle_\Lambda \tag{1.7}$$

where $e_0$ is a growing function of $g$, the expectation value is determined by $\rho_g$ and

$$Q_{g,\Gamma}(q_\Lambda) = \sum_{\langle x,x'\rangle\in\Gamma} Q_g(q_x, q_y)$$

$$Q_g(q_x, q_y) = \frac{1}{e_0}\left\{(q_x - q_{x'})^2 + \tfrac{4}{3}(|q_x^2 - e_0^2| + |q_{x'}^2 - e_0^2|)\right\}.$$

Here we have used the inequality

$$\chi^+(q_x)\,\chi^-(q_{x'}) \leqslant 4\sqrt{2\pi e e_0}\,\mathrm{e}^{-e_0}\exp\{Q_g(q_x, q_y)]\} \qquad e_0 \geqslant 1. \tag{1.8}$$

Theorem 1.1 will be proved if we prove the following lemma.

**Lemma 1.2.** *Let the conditions of theorem 1.1 be satisfied. Also let $e_0$ be given by (1.5). Then there exists a bounded function $E_0(g)$ on the interval $(1,\infty)$ such that*

$$\langle \mathrm{e}^{Q_{g,\Gamma}}\rangle \leqslant \mathrm{e}^{|\Gamma| E_0}. \tag{1.9}$$

In the next section we will give the proof of this lemma. Proofs of lemma 1.1 and equation (1.8) are standard (see [AKR, FL, GJS]). For the convenience of readers we give the proof of (1.8) which is easier to read since it is adapted to simpler systems.

## 2. Lemma 1.2 via a superstability argument

Changing the variables $q_x \to q_x - e_0$ in the integral in the right-hand side of (1.7) and using the translation invariance of the Lebesque measure we obtain

$$\begin{aligned} &\langle \mathrm{e}^{Q_{g,\Gamma}}\rangle = \int \rho(q_\Gamma + e_0)\exp\{Q_{g,\Gamma}(q_\Gamma + e_0)\}\,\mathrm{d}q_\Gamma \qquad q_\Gamma = (q_x, q_y;\ \langle x, y\rangle \in \Gamma) \\ &Q_{g,\Gamma}(q_\Gamma + e_0) \leqslant \sum_{\langle x,x'\rangle\in\Gamma}\left[\frac{10}{3e_0}(q_x^2 + q_{x'}^2) + \frac{8}{3}(|q_x| + |q_{x'}|)\right]. \end{aligned} \tag{2.1}$$

The polynomial $Q$ becomes bounded in $g$ if it is translated by $e_0$. As a result, we have to prove that the correlation functions, translated by $e_0$, in the limit of growing $g$ satisfy the usual superstability bound.

It is not difficult to check that if $e_0$ is given by (1.5) then

$$u_g^0(q) = 2d(\eta g^{-n} q^{2n} - q^2) = 2dn^{-1}[e_0^{-2n+2} q^{2n} - nq^2].$$

From this we immediately deduce that

$$u_g^0(q + e_0) = p_g(q) + bq^2 - b' \qquad b = 2dn^{-1}(n(2n-1) - n)$$

$$b' = 2d\frac{n-1}{n}e_0^2$$

where $p_g$ is a bounded-from-below polynomial in $e_0^{-1}$ and $q$ (the linear term proportional to $e_0$ is absent)

$$p_g(q) = 2dn^{-1}\sum_{s=3}^{2n}\frac{s!(2n-s)!}{n!}q^s e_0^{2-s}.$$

The following equalities are true:

$$g^{-s/2}e_0^s = (\eta n)^{-s/2(n-1)}g^{-N}$$

$$N = \frac{1}{2}\left[s - \frac{sn}{n-1}\right] = \frac{1}{2(n-1)}[s(n-1) - sn] = -\frac{s}{2(n-1)}$$

$$g^{-s/2}e_0^{s-1} = (\eta n)^{-(s-1)/2(n-1)}g^{-N}$$

$$N = \frac{1}{2}\left[s - \frac{(s-1)n}{n-1}\right] = \frac{1}{2(n-1)}[s(n-1) - (s-1)n] = \frac{n-s}{2(n-1)}$$

$$g^{-n/2}e_0^{n-1} = (\eta n)^{-1/2}.$$

Expanding $g^{-s/2}(q+e_0)^s$ in powers of $e_0$ we see that the term $g^{-s/2}e_0^s$ diverges and the other terms tend to zero for $s < n$ as $g$ tends to infinity. For $s = n$ the only term that survives is the term in which $e_0$ has the power $n-1$, i.e. the term linear in $q$. So, we have proved the following proposition.

**Proposition 2.1.** *The following equalities are valid:*

$$\lim_{g^{-1}\to 0}(u_g^0(q+e_0) + b') = bq^2$$

$$\lim_{g^{-1}\to 0}(u_g^1(q+e_0) - u_g^1(e_0)) = 0 \qquad n = 2k+1$$

$$\lim_{g^{-1}\to 0}(u_g^1(q+e_0) - u_g^1(e_0)) = u_{,n}^1 q \qquad n = 2k.$$

Now we have to establish the accurate superstability and regularity conditions for the potential energy translated by $e_0$.

First, we consider the potential energy (1.2).

$$U_g(q_X + e_0) = \sum_{x\in X} u_g(q+e_0) + \tfrac{1}{2}\sum_{\substack{|x-y|=1\\ x,y\in X}}(q_x - q_y)^2 + U_g'(q_X). \tag{2.2}$$

From the condition of theorem 1.1 we derive, taking into account the bound $(q_x - q_y)^2 \leqslant 2(q_x^2 + q_y^2)$

$$U_g(q_X + e_0) \geqslant \sum_{x\in X}\tilde{u}_g(q_x) - B_g(X) \tag{2.3}$$

where

$$\tilde{u}_g(q) = (u_g(q+e_0) + b' - u_g^1(e_0)) - Bv_g^0(q) \qquad v_g^0(q) = v^0(g^{-1/2}q)$$

$$B_g(X) = B_g|X| \qquad B_g = b' + B' - u_g^1(e_0).$$

For non-negative $U'$ (2.3) holds with

$$\tilde{u}_g(q) = u_g(q+e_0) + b' - u_g^1(e_0).$$

Let us put

$$\begin{aligned} U_{*g}(q_X) &= U_g(q_X + e_0) - \sum_{x\in\Lambda} u_{*g}(q_x) + B_g(X) \\ u_{*g} &= \tilde{u}_g - v_g \qquad v_g(q) = q^2 + v_g^0(q). \end{aligned} \tag{2.4}$$

Then the following superstability condition holds:

$$U_{*g}(q_X) \geqslant \sum_{x\in X} v_g(q_x). \tag{2.5}$$

The regularity condition also holds

$$|W_{*g}(q_{X_1}; q_{X_2})| = |U_{*g}(q_{X_1\cup X_2}) - U_{*g}(q_{X_1}) - U_{*g}(q_{X_2})|$$
$$\leqslant \frac{1}{2}\sum_{\substack{x\in X_1\\ y\in X_2}} \Psi_{|x-y|}[v_g(q_x) + v_g(q_y)] \qquad X_1 \cap X_2 = \emptyset \tag{2.6}$$

where $\Psi_{|x|} = 2\delta_{|x|,1} + \Psi'_{|x|}$.

Inequality (2.6) is derived from the equality

$$W_{*g}(q_{X_1}; q_{X_2}) = W'_g(q_{X_1}; q_{X_2}) + \frac{1}{2}\sum_{\substack{|x-y|=1\\ x\in X_1, y\in X_2}} (q_x - q_y)^2.$$

We have used, once more, the inequality $(q_x - q_y)^2 \leqslant 2(q_x^2 + q_y^2)$.

Applying the equalities $|X| - 1$ times, following from the regularity condition (2.6),

$$U_{*g}(q_x, q_X) \leqslant U_{*g}(q_x) + U_{*g}(q_X) + |W_{*g}(q_x; q_X)|$$
$$\leqslant U_{*g}(q_x) + U_{*g}(q_X) + \|\Psi\|_1 v_g(q_x) + \sum_{y\in X} \Psi_{|x-y|} v_g(q_y)$$

$$\sum_{x\in X}\sum_{y\in X'\in X} \Psi_{|x-y|} v_g(q_y) \leqslant \|\Psi\|_1 \sum_{y\in X'} v_g(q_y)$$

we obtain

$$U_{*g}(q_X) \leqslant \sum_{x\in X} \tilde{U}_g(q_x) \qquad \tilde{U}_g(q) = U_{*g}(q) + \|\Psi\|_1 v_g(q) \tag{2.7}$$

where

$$U_{*g}(q) = u_g(q + e_0) + b' - u_g^1(e_0) + B' - u_{*g}(q) = B' + Bv_g^0(q) + v_g(q).$$

For a positive $U'$ the term $Bv_g^0(q)$ has to be dropped in the last relation.

All the inequalities (2.5)–(2.7) are important for the derivation of the superstability bound for the correlation functions.

The analogues of (2.5)–(2.7) can be obtained for (1.1) by redefining slightly the above functions, putting for $X \subseteq \Lambda$

$$U_g(q_X) = \sum_{x\in X} u_g(q) + \frac{1}{2}\sum_{\substack{x,y\in X\\ |x-y|=1}} (q_x - q_y)^2 + U'_g(q_\Lambda) - (2d)^{-1}\sum_{x\in\partial\Lambda\cap X} u_g(q_x). \tag{2.8}$$

It is easy to check, taking into account the inequality

$$(2d)^{-1}\left|\sum_{x\in\partial\Lambda\cap X} (u_g(q_x + e_0) + b' - u_g^1(e_0))\right| \leqslant (2d)^{-1}\sum_{x\in X} |u_g^0(q_x + e_0) + b' - u_g^1(e_0)|$$

that (2.5)–(2.7) will hold if in (2.4) we put

$$B_g(X) = B_g|X| + (2d)^{-1}|X \cap \partial\Lambda|[b' - u_g^1(e_0)] \tag{2.9}$$

and add to the previous $\tilde{u}_g(q)$, $\tilde{U}_g(q)(U_{*g}(q))$ the terms

$$-(2d)^{-1}|u_g(q + e_0) + b' - u_g^1(e_0)| \qquad d^{-1}|u_g(q + e_) + b' - u_g^1(e_0)|$$

respectively.

We see that the regularity condition is the same for both cases and $v_g(q)$ is not changed.

$B_g(\Lambda)$ diverges if $g$ tends to infinity since $b', u_g^1(e_0)$ diverge. We can add $B_g(\Lambda)$ to the potential energy since the expression for the correlation functions is not changed after this.

Let us put

$$\rho_{*g}^{\Lambda}(q_\Lambda) = \exp\left\{\beta \sum_{x\in\Lambda} u_{*g}(q_x)\right\}\rho_g^{\Lambda}(q_\Lambda + e_0).$$

Then $\rho_{*g}^{\Lambda}$ are expressed in terms of $U_{*g}$ after adding to $U_g$ the terms large in $g$ independent of oscillator variables

$$\begin{aligned} \rho_{*g}^{\Lambda}(q_X) &= Z_{*\Lambda}^{-1}\int \mathrm{e}^{-\beta U_{*g}(q_\Lambda)}\mu_{*g}(\mathrm{d}q_{\Lambda\setminus X}) \\ Z_{*\Lambda} &= \int \mathrm{e}^{-\beta U_{*g}(q_\Lambda)}\mu_*(\mathrm{d}q_\Lambda) \end{aligned} \tag{2.10}$$

where

$$\mu_{*g}(\mathrm{d}q_Y) = \exp\left\{-\beta \sum_{x\in Y} u_{*g}(q_x)\right\}\mathrm{d}q_Y.$$

As a result of the superstability and regularity conditions for $U_{*g}$ the following theorem is true [R].

**Theorem 2.1.** *Let the conditions (2.5) and (2.6) hold for a positive polynomial* $v_g$ *and* $u_{*g}$ *such that the measure* $\mu_{*g}$ *is finite. Then for arbitrary* $0 < 3\varepsilon < 1$, $r > 0$ *for the correlation functions defined by (2.10), the following (superstability) bound is valid:*

$$\rho_{*g}^{\Lambda}(q_X) \leqslant \exp\left\{-\sum_{x\in X}\left[\beta(1-3\varepsilon)v_g(q_x) - c(I_{0u_{*g}}^{-1}, I_{u_{*g}})\right]\right\} \tag{2.11}$$

*where* $c$ *is a continuous function,*

$$I_{0u} = \int_{|q|\leqslant r} \exp\{-\beta[\tilde{U}_g(q) + u(q)]\}\,\mathrm{d}q$$

$$I_u = \int \exp\{-\beta[(1-3\varepsilon)v_g(q) + u(q)]\}\,\mathrm{d}q$$

*where* $\tilde{U}_g$ *is given by (2.7).*

We formulated the Ruelle theorem in a special form in order to indicate the dependence of $c$ on $g$ ($c$ also depends on $\|\Psi\|_1, r$).

Equation (2.1) and theorem 2.1 yields

$$\begin{aligned} &\langle \mathrm{e}^{Q_{g,\Gamma}}\rangle \leqslant \mathrm{e}^{|\Gamma|E_0} \qquad E_0 = E^0 + e_*(g) \qquad e_*(g) = 2c\big(I_{0u_{*g}}^{-1}, I_{u_{*g}}\big) \\ &E^0 = 2\ln\int \exp\left\{-\beta(1-3\varepsilon)v_g(q) - \beta u_{*g}(q) + \frac{10}{3e_0}q^2 + \frac{8}{3}|q|\right\}\mathrm{d}q. \end{aligned} \tag{2.12}$$

It is clear that

$$(1-3\varepsilon)v_g(q) + u_{*g}(q) = \tilde{u}_g(q) - 3\varepsilon v_g(q).$$

As a result, equation (1.3) holds with $E$ given by (1.5). From the conditions of theorem 1.1 it follows that $E^0$ and $e_*$ exist in the limit of vanishing $g^{-1}$, since $\tilde{u}_g(q) - 3\varepsilon v_g(q)$ tends to the strictly positive quadratic polynomial that is necessary for existence of $E^0$, $I_{u_{*g}}$. Here proposition 2.1 plays a significant role and the following relations are used:

$$\lim_{g^{-1}\to 0} \tilde{u}_g(q) = bq^2 \qquad \lim_{g^{-1}\to 0} v_g(q) = kq^2$$

$$b = 4d(n-1) > 8 \qquad 3\varepsilon < 1 \qquad (1-(2d)^{-1}) > (1-d^{-1})b > 4$$

where $k = 1$ for $U'$ non-positive and $k = 2$ for $U'$ positive.

The functions $I_u$, $\mathrm{e}^{E^0/2}$ are uniform limits of sequences of continuous bounded functions which are integrals over intervals $[0, n]$; that is, they are also continuous and bounded. Lemma 1.3 is proved. Application of lemma 1.1 completes the proof of theorem 1.1.

**Proof of (1.8).** Let $e_0 = \lambda^{-1} \geqslant 1$.

$$\begin{aligned}\chi_x^+\chi_{x'}^- &= [\chi_{(0,1/2\lambda)}(q_x) + \chi_{(1/2\lambda,\infty)}(q_x)][\chi_{(-\infty,-1/2\lambda)}(q_{x'}) + \chi_{(-1/2\lambda,0)}(q_{x'})] \\ &\leqslant \chi_{(1/2\lambda,\infty)}(q_x)\chi_{(-\infty,-1/2\lambda)}(q_{x'}) + \chi_{(0,1/2\lambda)}(q_x)\chi_{(-1/2\lambda,0)}(q_{x'}) \\ &\quad + \chi_{(0,1/2\lambda)}(q_x) + \chi_{(-1/2\lambda,0)}(q_{x'}).\end{aligned} \tag{2.13}$$

If $q_x \geqslant 1/2\lambda$, $q_{x'} \leqslant -1/2\lambda$ then $\lambda(q_x - q_{x'}) \geqslant 1$ and for an arbitrary integer $M$ $[\lambda^2(q_x - q_{x'})^2]^M \geqslant 1$. As a result of this inequality and the Cauchy formula

$$\begin{aligned}\chi_{(1/2\lambda,\infty)}(q_x)\chi_{(-\infty,-1/2\lambda)}(q_x) &\leqslant [\lambda^2(q_x - q_{x'})^2]^M \leqslant \lambda^M \left\{\frac{\mathrm{d}^M}{\mathrm{d}\xi^M}\mathrm{e}^{\lambda\xi(q_x-q_{x'})^2}\right\}(\xi = 0) \\ &= \frac{M!}{2\pi \mathrm{i}}\lambda^M \int_{|\xi|=1} \mathrm{e}^{\lambda\xi(q_x-q_{x'})^2}\frac{\mathrm{d}\xi}{\xi^{M+1}}.\end{aligned}$$

Taking the absolute value of the last integral and passing to polar coordinates ($\mathrm{d}\xi$ is treated as the first-order differential form on $S^1$) we derive the bound

$$\chi_{(1/2\lambda,\infty)}(q_x)\chi_{(-\infty,-1/2\lambda)}(q_x) \leqslant M!\,\lambda^M \mathrm{e}^{\lambda(q_x-q_{x'})^2}.$$

In the same way we obtain

$$\begin{aligned}\chi_{(0,1/2\lambda)}(q_x) &\leqslant \left[\tfrac{4}{3}(1-\lambda^2 q_x^2)\right]^M = \lambda^M\left\{\frac{\mathrm{d}^M}{\mathrm{d}\xi^M}\mathrm{e}^{\frac{4}{3}\lambda\xi(\lambda^{-2}-q_x^2)}\right\}(\xi = 0) \\ &= \frac{M!}{2\pi \mathrm{i}}\lambda^M \int_{|\xi|=1} \mathrm{e}^{\frac{4}{3}\lambda\xi(\lambda^{-2}-q_x^2)}\frac{\mathrm{d}\xi}{\xi^{M+1}}\end{aligned}$$

$$\chi_{(0,1/2\lambda)}(q_x) \leqslant M!\,\lambda^M \mathrm{e}^{\frac{4}{3}\lambda|q_x^2-\lambda^{-2}|}$$

$$\chi_{(-1/2\lambda,0)}(q_x) \leqslant [\tfrac{4}{3}(1-\lambda^2 q_x^2)]^M \leqslant M!\,\lambda^M \mathrm{e}^{\frac{4}{3}\lambda|q_x^2-\lambda^{-2}|}.$$

Substituting these inequalities into (2.13) we obtain

$$\chi_x^+\chi_{x'}^- \leqslant 4M!\,\lambda^M \mathrm{e}^{\lambda[(q_x-q_{x'})^2 + \frac{4}{3}(|q_x^2-\lambda^{-2}|+|q_{x'}^2-\lambda^{-2}|)]}. \tag{2.14}$$

Let $M = [\lambda^{-1}]$, i.e the integer part of $\lambda^{-1}$, then with the help of the inequalities

$$n! \leqslant \sqrt{2\pi n}\,(ne^{-1})^n \qquad M \leqslant \lambda^{-1}$$

it follows that

$$M!\,\lambda^M = \sqrt{2\pi M}(Me^{-1})^M\lambda^M \leqslant \sqrt{2\pi\lambda^{-1}}[\lambda^{-1}]^M\lambda^M\mathrm{e}^{-[\lambda^{-1}]} \leqslant \sqrt{2\pi e\lambda^{-1}}\mathrm{e}^{-\lambda^{-1}}.$$

Here we have used the fact that the difference of a number and its integer part does not exceed the unit and the inequality $\lambda^{-1} \geqslant 1$. Equation (1.8) is proved. □

**Remarks**

1. Lemma 1.1 is true for arbitrary dimension $d > 1$. Though its proof is given for $d = 2$ in [FL] the main idea of the proof remains the same in a general case.
2. If one adds to (1.1) and (1.2) the non-translation-invariant term $\sum_{x,y\in\Lambda,|x-y|>1} C_{x-y}q_xq_y$ then is not difficult to apply our arguments and prove that the conclusion of theorem 1.1 holds if

$$|C_{x-y}| \leqslant C^0_{|x-y|} > 0 \qquad \|C^0\|_1 < \infty.$$

3. The system characterized by (1.2) can be considered as basic for our approach. It is clear that it is very convenient to deal with and may be considered canonical, since the one-particle potential in (1.2) has minima which may be associated to pure phases of the Gibbs system.
   There is an interesting problem in proving theorem 1.1 for more general external potentials $u$. The polynomial $u^0_g$ has two real symmetric minima. So we expect that there are only two pure phases in the corresponding Gibbs system. $e_0$ is an approximate (asymptotic) minimum of $u$ from theorem 1.1 and this fact may mean that one needs to find such a minima for more general one-particle potentials instead of the *bona fide* minima.
4. Theorem 1.1 proves the existence of a phase transition for the case where $U'$ is expressed through a pair (special) potential, since it is known that in this case in the high-temperature phase there is an exponential decrease of correlations [K]. The (approximate) critical temperature may be determined from the bound (1.4) finding the temperature, depending on $g$, for which $a' = \infty$. It can be shown that there exists a constant $a_0 < 1$ such that this equality occurs if $\mathrm{e}^{-E} = a_0$. From this the critical temperature can be found.
5. The magnitude of NN interaction plays an exceptional role in the proposed approach since it vanishing automatically implies a vanishing of the spin two-point function for NN sites. This means that $E$ in (1.3) has to depend on the magnitude of the NN interaction, tending to zero together with it. So, one should always rescale by the magnitude (in an appropriate power) all the variables, when starting to derive the Peierls-type contour bound using (1.8) with $e_0$ depending on it.
6. Essentially ferromagnetic interaction may be characterized by the property that the ferromagnetic configuration, consisting of the coordinate $e_0$ (a minimum or an approximate minimum of a one-particle potential) at each lattice site, is more favourable than the associated antiferromagnetic (staggered) configuration, consisting of the coordinate $e_0$ at the even sublattice and $-e_0$ at the odd sublattice for sufficiently large $g$, i.e. the potential energy on the former configuration is less than on the latter. This property follows from the superstability condition for the rescaled $U'_g$ in the formulation of theorem 1.1 and the fact that the growth in $g$ of $g^{-s}e_0^{2s}$, $s < n$, is slower than $e_0^2$ (see proposition 2.1). In other words, the ferromagnetic NN part of the potential energy suppresses antiferromagnetic ground states for sufficiently large $g$.
   We believe that ferromagnetic LRO occurs in systems that are slightly more general than essentially ferromagnetic.

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